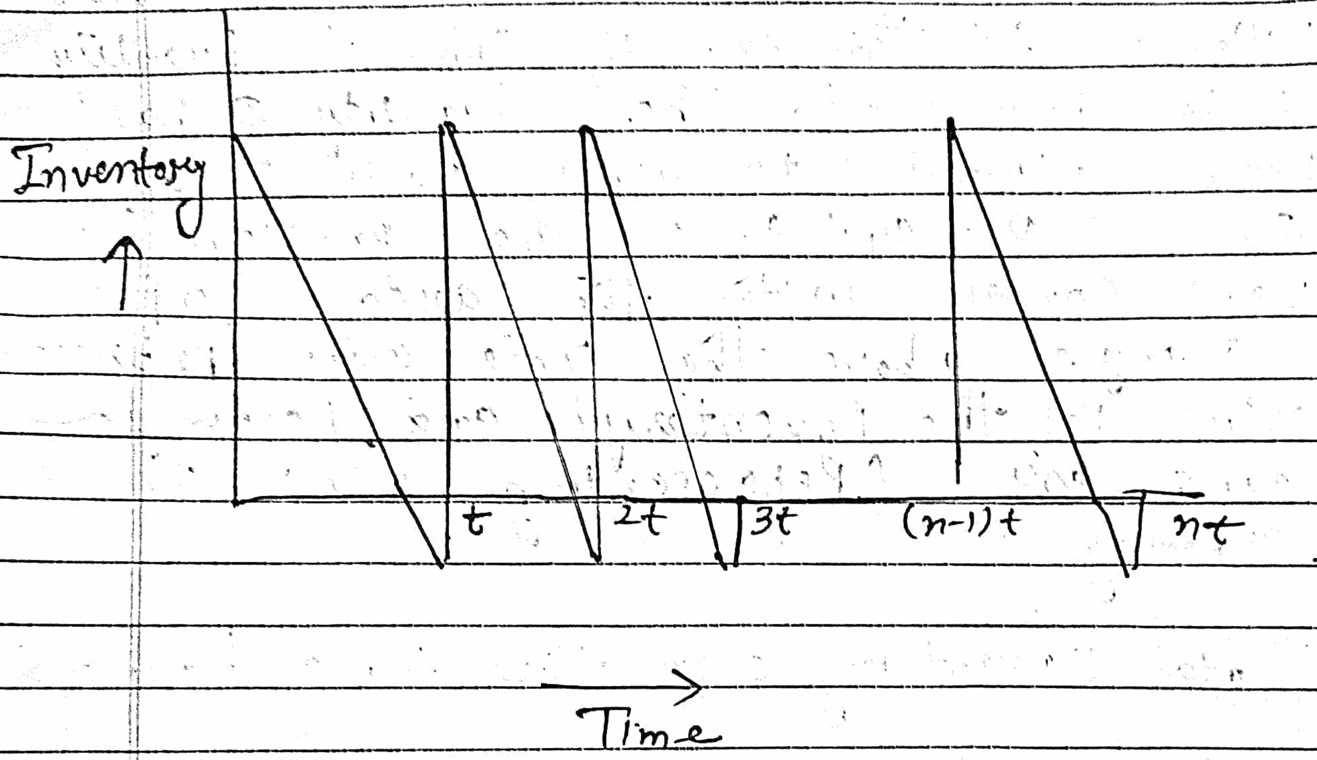
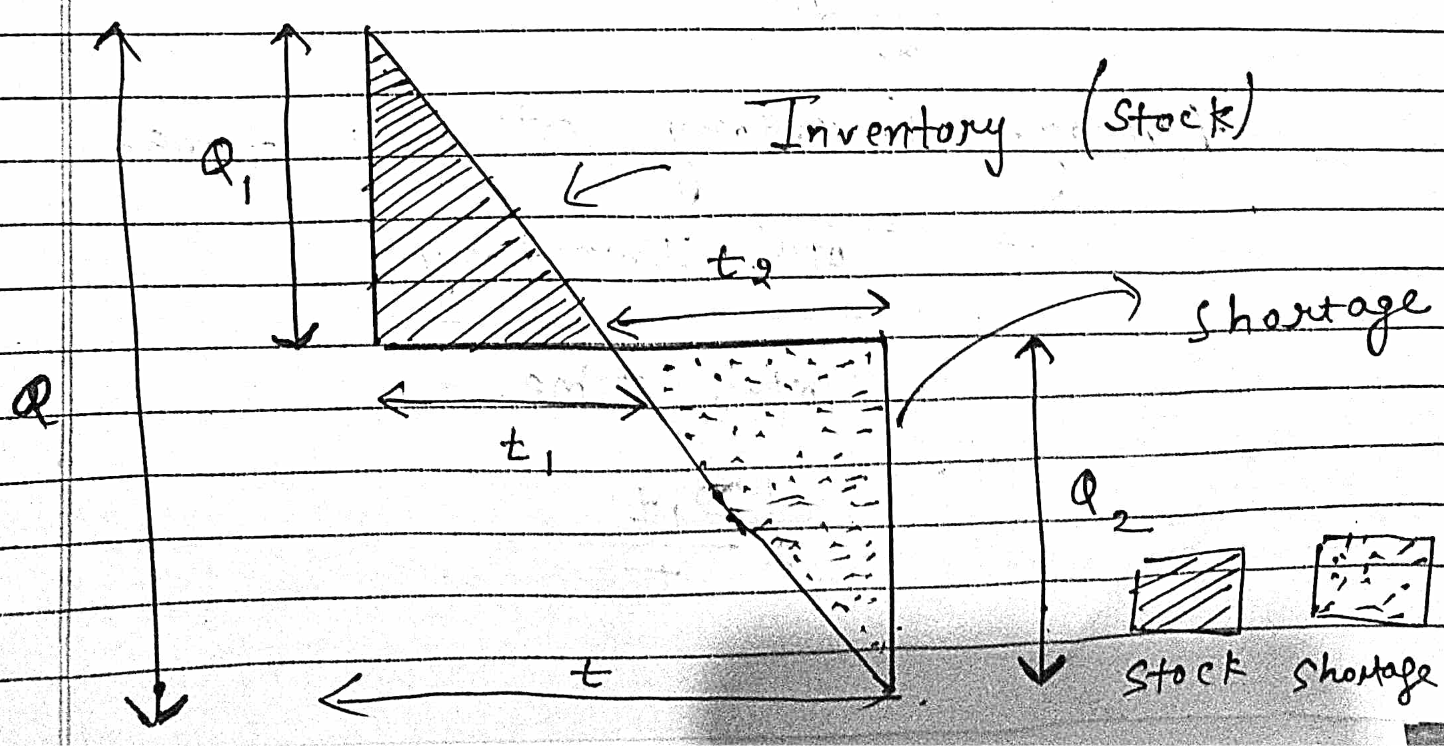
Problem of EOQ with Shortages

Here Shortages are permitted. Let C_2 be the Shortages cost per unit of time per unit quantity. The inventory situation can be illustrated graphically as shown below





Here the total time period is 1 year which is divided into equal parts say of interval t. Further the time period t is divided into two parts t_1 and t_2 i.e $t = t_1 + t_2$. During the time Interval t_1 the items are drawn from the inventory as needed. During the time t_2 orders able take but not Filled. At the end of time t Quantity Q is produced. The Quantity Q has been divided into two parts Q_1 and Q_2 i.e $Q = Q_1 + Q_2$. The problem is now concern with the area of triangle above the time axis (Representing items in the inventory) and below the same axis (Representing items in shortages)

Total Inventory over the time period

$$t = \frac{1}{2} Q_1 t_1$$ { Area of Δ $= \frac{1}{2} \times Base \times Height$ }

Average inventory at any time

$$= \frac{\text{Total Inventory}}{\text{Total time}}$$

$$\frac{\frac{1}{2} Q_1 t_1}{t}$$

Annual Inventory holding cost

$= \frac{1}{2} \frac{Q_1 t_1}{t} C_1$ [C_1 = Holding cost per unit]

Total Amount of Shortages Over the time period t $= \frac{1}{2} Q_2 t_2$

Average Shortage at any time

$= \frac{1}{2} \frac{Q_2 t_2}{t}$ [$\rightarrow$ $\frac{\text{Total Inventory}}{\text{Total time}}$

Annual Shortage cost $= \frac{1}{2} Q_2 t_2 C_2$

Annual Cost associated with runs of size $Q = n C_s$

But: $D = nQ$

$\frac{D}{Q} = n$

Put value of n

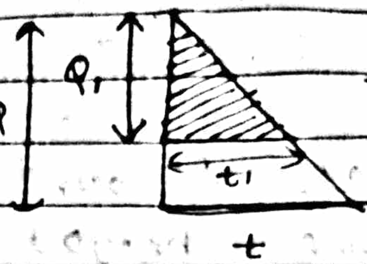
Annual cost Associated with runs of Size of Q $= \frac{D}{Q} C_s$

Total cost $C_A = f(Q) + g(Q)$

$$C_1 \frac{1}{2} \frac{Q_1 t_1}{t} + C_2 \left(\frac{1}{2} \frac{Q_2 t_2}{t}\right) + \frac{D}{Q} C_s \quad \text{--- (1)}$$

Now using the Relationship for similar triangle

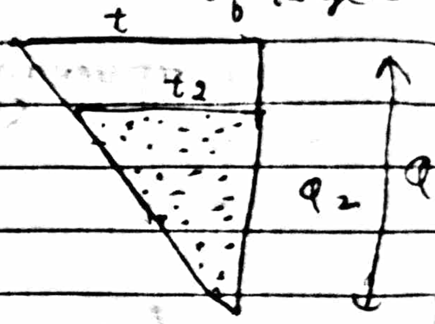
$$\frac{t_1}{t} = \frac{\varphi_1}{\varphi}$$



$$\frac{\text{Base of small } \Delta}{\text{Base of Large } \Delta} = \frac{\text{Perpendicular of small } \Delta}{\text{Perpendicular of large } \Delta}$$

$$\boxed{t_1 = \frac{\varphi_1 t}{\varphi}}$$

and $$\frac{t_2}{t} = \frac{\varphi_2}{\varphi}$$



$$\boxed{t_2 = \frac{\varphi_2 t}{\varphi}}$$

put value of t_1 and t_2 in eq (1)

$$C_A = \left[c_1 \frac{1}{2} \frac{\varphi_1}{\cancel{t}} \left(\frac{\varphi_1 \cancel{t}}{\varphi}\right) + c_2 \left(\frac{1}{2} \frac{\varphi_2}{t} \left(\frac{\varphi_2 t}{\varphi}\right)\right)\right] + \frac{D}{\varphi} c_3$$

$$C_A = \left[c_1 \frac{1}{2} \frac{\varphi_1^2}{\varphi} + c_2 \left(\frac{1}{2} \frac{\varphi_2^2}{\varphi}\right)\right] + \frac{D}{\varphi} c_3$$

Convert all terms int φ_1 By using Relation

$$\varphi = \varphi_1 + \varphi_2$$
$$\varphi_2 = \varphi - \varphi_1$$

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$$C_A = \left[c_1 \frac{1}{2} \frac{Q_1^2}{Q} + c_2 \left(\frac{1}{2} \cdot \frac{(Q - Q_1)^2}{Q} \right) \right] + \frac{DC_3}{Q}$$

Now Differentiate partially w.r.t Q_1 and put $\frac{\partial C_A}{\partial Q_1} = 0$ to find value of Q_1

$$\frac{\partial C_A}{\partial Q_1} = \frac{\partial}{\partial Q_1} \left\{ \left[c_1 \frac{1}{2} \frac{Q_1^2}{Q} + c_2 \left(\frac{1}{2} \frac{(Q - Q_1)^2}{Q} \right) \right] + \frac{DC_3}{Q} \right\}$$

$$\frac{\partial C_A}{\partial Q_1} = \frac{\partial}{\partial Q_1} c_1 \frac{1}{2} \frac{Q_1^2}{Q} + \frac{\partial}{\partial Q_1} c_2 \left(\frac{1}{2} \frac{(Q - Q_1)^2}{Q} \right) + \frac{\partial}{\partial Q_1} \frac{DC_3}{Q}$$

$$\frac{\partial C_A}{\partial Q_1} = \frac{\cancel{2} c_1 Q_1}{\cancel{2} Q} + \frac{1 \times \cancel{2} (-1)(Q - Q_1) c_2}{\cancel{2} Q} + 0$$

$$\frac{\partial C_A}{\partial Q_1} = \frac{c_1 Q_1}{Q} - \frac{(Q - Q_1)}{Q} c_2$$

Put $\frac{\partial C_A}{\partial Q_1} = 0$

$$\frac{c_1 Q_1}{Q} - \frac{(Q - Q_1) c_2}{Q} = 0$$

$$\frac{c_1 Q_1 - (Q - Q_1) c_2}{Q} = 0$$

$$c_1 Q_1 - (Q - Q_1) c_2 = 0$$

$$c_1 Q_1 - c_2 Q + c_2 Q_1 = 0$$

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$$c_1 Q_1 + c_2 Q_1 - c_2 Q = 0$$

$$(c_1 + c_2) Q_1 - c_2 Q = 0$$

$$(c_1 + c_2) Q_1 = c_2 Q$$

$$\boxed{Q_1 = \frac{c_2 Q}{c_1 + c_2}}$$

Now we have to find value of Q

from (*)

$$C_A = \left[c_1 \frac{1}{2} \frac{Q_1^2}{Q} + c_2 \left(\frac{1}{2} \frac{(Q - Q_1)^2}{Q} \right) \right] + \frac{D}{Q} c_3$$

$$C_A = \left[c_1 \frac{1}{2} \frac{Q_1^2}{Q} + c_2 \left(\frac{1}{2} \frac{(Q^2 + Q_1^2 - 2QQ_1)}{Q} \right) \right] + \frac{D}{Q} c_3$$

$$C_A = \left[c_1 \frac{1}{2} \frac{Q_1^2}{Q} + c_2 \left(\frac{1}{2} \left(\frac{Q^2}{Q} + \frac{Q_1^2}{Q} - \frac{2QQ_1}{Q} \right) \right) \right] + \frac{D}{Q} c_3$$

$$C_A = \left[c_1 \frac{1}{2} \frac{Q_1^2}{Q} + c_2 \left(\frac{1}{2} \left(Q + \frac{Q_1^2}{Q} - 2Q_1 \right) \right) \right] + \frac{D}{Q} c_3$$

$$C_A = \left[c_1 \frac{1}{2} \frac{Q_1^2}{Q} + c_2 \left(\frac{1}{2} \left(Q + Q_1^2 Q^{-1} - 2Q_1 \right) \right) \right] + DQ^{-1} c_3$$

$$C_A = c_1 \frac{1}{2} Q_1^2 Q^{-1} + c_2 \left(\frac{1}{2} \left(Q + Q_1^2 Q^{-1} - 2Q_1 \right) \right) + DQ^{-1} C_3$$

Differentiate both sides partialy w.r.t Q

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$$\frac{\partial CA}{\partial Q} = \frac{\partial}{\partial Q}\left[c_1 \frac{1}{2} Q_1^2 Q^{-1} + c_2\left(\frac{1}{2}\left(Q + Q_1^2 Q^{-1} - 2Q_1\right)\right]\right.$$

$$+ \frac{\partial}{\partial Q} D Q^{-1} C_s$$

$$\frac{\partial}{\partial Q} c_1 \frac{1}{2} Q_1^2 Q^{-1} + \frac{\partial}{\partial Q} c_2\left(\frac{1}{2}\left(Q + Q_1^2 Q^{-1} - 2Q_1\right)\right)$$

$$+ (-1) D Q^{-2} C_s$$

$$\frac{1}{2} \times c_1 (-1) Q_1^2 (Q)^{-2} + c_2\left(\frac{1}{2}\left(1 + (-1) Q_1^2 Q^{-2} - 0\right)\right)$$

$$- D Q^{-2} C_s$$

$$\frac{-c_1 Q_1^2 Q^{-2}}{2} + c_2\left(\frac{1}{2}\left(1 - Q_1^2 Q^{-2}\right)\right)$$

$$- D Q^{-2} C_s$$

$$= \frac{-c_1 Q_1^2}{2Q^2} + c_2\left(\frac{1}{2}\left(1 - \frac{Q_1^2}{Q^2}\right)\right) - \frac{D C_s}{Q^2}$$

$$= \frac{-c_1 Q_1^2}{2Q^2} + c_2\left(\frac{1}{2}\left(\frac{Q^2 - Q_1^2}{Q^2}\right)\right) - \frac{D C_s}{Q^2}$$

$$= -\frac{c_1 Q_1^2}{2Q^2} + \frac{c_2}{2Q^2}\left(Q^2 - Q_1^2\right) - \frac{D C_s}{Q^2}$$

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put $\frac{\partial CA}{\partial Q} = 0$ to find value of Q

$$\frac{-c_1 Q_1^2}{2Q^2} + \frac{c_2 (Q^2 - Q_1^2)}{2Q^2} - \frac{D c_3}{Q^2} = 0$$

$$\frac{-c_1 Q_1^2 + c_2 (Q^2 - Q_1^2) - 2 D c_3}{2Q^2} = 0$$

$$-c_1 Q_1^2 + c_2 (Q^2 - Q_1^2) - 2 D c_3 = 0$$

$$-c_1 Q_1^2 + c_2 Q^2 - c_2 Q_1^2 - 2 D c_3 = 0$$

$$-c_1 Q_1^2 - c_2 Q_1^2 + c_2 Q^2 - 2 D c_3 = 0$$

$$-(c_1 + c_2) Q_1^2 + c_2 Q^2 - 2 D c_3 = 0$$

$$c_2 Q^2 = 2 D c_3 + (c_1 + c_2) Q_1^2$$

$$Q^2 = \frac{2 D c_3 + (c_1 + c_2) Q_1^2}{c_2}$$

$$Q^2 = \frac{2 D c_3}{c_2} + \left(\frac{c_1 + c_2}{c_2}\right) Q_1^2$$

$$Q^2 = \frac{2 D c_3}{c_2} + \left(\frac{c_1}{c_2} + 1\right) Q_1^2$$

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$$Q^2 = \frac{2DC_3}{C_2} + \frac{C_1}{C_2}Q_1^2 + Q_1^2$$

$$Q^2 = \frac{2DC_3 + C_1 Q_1^2}{C_2} + Q_1^2$$

$$Q = \sqrt{\frac{2DC_3 + C_1 Q_1^2}{C_2} + Q_1^2}$$

Also $\frac{\partial^2 CA}{\partial Q^2} > 0$ and $\frac{\partial^2 CA}{\partial Q_1^2} > 0$

Also $Q_1 = \frac{C_2 Q}{C_1 + C_2}$

put value of Q in this equation

$$Q_1 = \frac{C_2}{C_1 + C_2}\sqrt{\frac{2DC_3 + C_1 Q_1^2}{C_2} + Q_1^2}$$

$$\frac{C_1 + C_2}{C_2} Q_1 = \sqrt{\frac{2DC_3 + C_1 Q_1^2}{C_2} + Q_1^2}$$

Squaring Both sides

$$\frac{(C_1 + C_2)^2}{C_2^2} Q_1^2 = \left(\sqrt{\frac{2DC_3 + C_1 Q_1^2}{C_2} + Q_1^2}\right)^2$$

$$\frac{(C_1 + C_2)^2}{C_2^2} Q_1^2 = \frac{2DC_3 + C_1 Q_1^2}{C_2} + Q_1^2$$

$$\frac{(C_1 + C_2)^2}{\cancel{C_2^2}} Q_1^2 = \frac{2DC_3 + C_1 Q_1^2 + C_2 Q_1^2}{\cancel{C_2}}$$

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$$\frac{(c_1+c_2)^2 Q_1^2}{c_2} = 2Dc_3 + c_1 Q_1^2 + c_2 Q_1^2$$

Taking Q_1^2 common

$$\frac{(c_1+c_2)^2 \cancel{Q_1^2}}{c_2} = \cancel{Q_1^2}\left[\frac{2Dc_3}{Q_1^2} + c_1 + c_2\right]$$

$$\frac{(c_1+c_2)^2}{c_2} = \left[\frac{2Dc_3}{Q_1^2} + c_1 + c_2\right]$$

$$\frac{c_1^2+c_2^2+2c_1c_2}{c_2} = \frac{2Dc_3}{Q_1^2} + c_1 + c_2$$

$$\frac{c_1^2+c_2^2+2c_1c_2}{c_2} - c_1 - c_2 = \frac{2Dc_3}{Q_1^2}$$

$$\frac{c_1^2+\cancel{c_2^2}+2c_1c_2 - c_1c_2 - \cancel{c_2^2}}{c_2} = \frac{2Dc_3}{Q_1^2}$$

$$\frac{c_1^2 + 2c_1c_2 - c_1c_2}{c_2} = \frac{2Dc_3}{Q_1^2}$$

$$\frac{c_1^2 + c_1c_2}{c_2} = \frac{2Dc_3}{Q_1^2}$$

$$\frac{c_1^2 + c_1c_2}{c_2} \times \frac{1}{2Dc_3} = \frac{1}{Q_1^2}$$

$$Q_1^2 = \frac{2DC_3C_2}{C_1^2 + C_1C_2}$$

$$Q_1^2 = \frac{2DC_2C_3}{C_1(C_1+C_2)}$$

$$Q_1 = \sqrt{\frac{2DC_3C_2}{C_1(C_1+C_2)}}$$

$$Q_1 = \sqrt{\frac{C_2}{C_1+C_2}}\sqrt{\frac{2DC_3}{C_1}}$$

Now $$Q^\circ = \sqrt{\frac{2DC_3 + C_1Q_1^{\circ 2}}{C_2} + Q_1^{\circ 2}}$$

$$Q^\circ = \sqrt{\frac{2DC_3}{C_2} + \frac{C_1Q_1^{\circ 2}}{C_2} + Q_1^{\circ 2}}$$

Put value of $Q_1^{\circ 2}$

$$Q^\circ = \sqrt{\frac{2DC_3}{C_2} + \frac{\cancel{C_1}}{\cancel{C_2}} \times \frac{\cancel{C_2}}{C_1+C_2} \times \frac{2DC_3}{\cancel{C_1}} + \frac{C_2}{C_1+C_2} \times \frac{2DC_3}{C_1}}$$

$$Q^\circ = \sqrt{\frac{2DC_3}{C_2} + \frac{2DC_3}{C_1+C_2} + \frac{2DC_3C_2}{C_1(C_1+C_2)}}$$

$$Q^0 = \sqrt{\frac{2DC_3}{C_1}\left(\frac{C_1}{C_2} + \frac{C_1}{C_1+C_2} + \frac{C_2}{C_1+C_2}\right)}$$

$$Q^0 = \sqrt{\frac{2DC_3}{C_1}\left(\frac{C_1(C_1+C_2) + C_1C_2 + C_2^2}{C_2(C_1+C_2)}\right)}$$

$$Q^0 = \sqrt{\frac{2DC_3}{C_1}\left[\frac{C_1^2 + C_1C_2 + C_1C_2 + C_2^2}{C_2(C_1+C_2)}\right]}$$

$$\sqrt{\frac{2DC_3}{C_1}\left[\frac{\cancel{(C_1+C_2)^2}}{C_2\cancel{(C_1+C_2)}}\right]}$$

$$\sqrt{\frac{2DC_3}{C_1}\frac{(C_1+C_2)}{C_2}}$$

$$\boxed{Q^0 = \sqrt{\frac{2DC_3(C_1+C_2)}{C_1C_2}}}$$

The optimum cost Equation is

$$C_A^0 = \frac{1}{2}C_1\frac{Q_1^{0\,2}}{Q^0} + \frac{1}{2}C_2\frac{(Q^0 - Q_1^0)^2}{Q^0} + \frac{D}{Q^0}C_3$$

$$C_A^\circ = \frac{1}{2Q^\circ} \cdot c_1 Q_1^{\circ 2} + \frac{c_2(Q^\circ - Q_1^\circ)^2}{2Q^\circ} + \frac{D}{Q^\circ} c_3$$

Taking $\frac{1}{2Q^\circ}$ common

$$C_A^\circ = \frac{1}{2Q^\circ}\left[c_1 Q_1^{\circ 2} + c_2(Q^\circ - Q_1^\circ)^2 + 2Dc_3\right]$$

$$C_A^\circ = \frac{1}{2Q^\circ}\left[c_1 Q_1^{\circ 2} + c_2 Q^{\circ 2}\left[1 - \frac{Q_1^\circ}{Q^\circ}\right]^2 + 2Dc_3\right]$$

put values of Q° and $Q_1^{\circ 2}$

$$C_A^\circ = \frac{1}{2}\sqrt{\frac{c_1 c_2}{2Dc_3(c_1+c_2)}}\left[\frac{\cancel{c_1} \times 2Dc_3 c_2}{(c_1+c_2)\cancel{c_1}} + \frac{\cancel{c_2}\, 2Dc_3(c_1+c_2)}{c_1 \cancel{c_2}}\right]\left[1 - \sqrt{\frac{2c_2 c_3 D}{c_1(c_1+c_2)}}\sqrt{\frac{c_1 c_2}{2Dc_3(c_1+c_2)}}\right]^2$$

$$+ 2Dc_3 .$$

$$C_A^\circ = \frac{1}{2}\sqrt{\frac{c_1 c_2}{2Dc_3(c_1+c_2)}}\left[\frac{2Dc_3 c_2}{(c_1+c_2)} + \frac{2Dc_3(c_1+c_2)}{c_1}\right.$$

$$\left(1 - \sqrt{\frac{\cancel{2}\, c_2 \cancel{c_3 D}}{\cancel{c_1}(c_1+c_2)} \times \frac{\cancel{c_1} c_2}{\cancel{2Dc_3}(c_1+c_2)}}\right)^2 + 2Dc_3$$

$$C_A^{\circ} = \frac{1}{2}\sqrt{\frac{c_1 c_2}{2Dc_3(c_1+c_2)}}\left[\frac{2Dc_3c_2}{c_1+c_2} + \frac{2Dc_3(c_1+c_2)}{c_1}\left(1-\sqrt{\frac{c_2^2}{(c_1+c_2)^2}}\right)^2 + 2Dc_3\right]$$

$$C_A^{\circ} = \frac{1}{2}\sqrt{\frac{c_1 c_2}{2Dc_3(c_1+c_2)}}\left[\frac{2Dc_3c_2}{c_1+c_2} + \frac{2Dc_3(c_1+c_2)}{c_1}\left(1-\frac{c_2}{c_1+c_2}\right)^2 + 2Dc_3\right]$$

$$C_A^{\circ} = \frac{1}{2}\sqrt{\frac{c_1 c_2}{2Dc_3(c_1+c_2)}}\, 2Dc_3\left[\frac{c_2}{c_1+c_2} + \frac{c_1+c_2}{c_1}\left(\frac{c_1+\cancel{c_2}-\cancel{c_2}}{c_1+c_2}\right)^2 + 1\right]$$

$$C_A^{\circ} = \frac{1}{2}\sqrt{\frac{c_1 c_2}{2Dc_3(c_1+c_2)}}\, 2Dc_3\left[\frac{c_2}{c_1+c_2} + \frac{\cancel{c_1+c_2}}{\cancel{c_1}}\left(\frac{\cancel{c_1^2}}{(\cancel{c_1+c_2})^2}\right) + 1\right]$$

$$\frac{1}{2}\sqrt{\frac{c_1c_2}{2Dc_3(c_1+c_2)}}\quad 2Dc_3\left[\frac{c_2}{c_1+c_2}+\frac{c_1}{c_1+c_2}+1\right]$$

$$\frac{1}{2}\sqrt{\frac{c_1c_2}{2Dc_3(c_1+c_2)}}\quad 2Dc_3\left[\cancel{\frac{c_1+c_2}{c_1+c_2}}+1\right]$$

$$\frac{1}{2}\sqrt{\frac{c_1c_2}{2Dc_3(c_1+c_2)}}\times 2Dc_3\,[1+1]$$

$$\frac{1}{2}\sqrt{\frac{c_1c_2}{2Dc_3(c_1+c_2)}}\times 2Dc_3\,(2)$$

$$\cancel{\frac{1}{2}\times 2}\times 2Dc_3\sqrt{\frac{c_1c_2}{2Dc_3(c_1+c_2)}}$$

$$c_A^{\circ} = 2Dc_3\sqrt{\frac{c_1c_2}{2Dc_3(c_1+c_2)}}$$

$$\sqrt{(2Dc_3)^2}\sqrt{\frac{c_1c_2}{2Dc_3(c_1+c_2)}}$$

$$c_A^{\circ} = \sqrt{4D^2c_3^2}\sqrt{\frac{c_1c_2}{2Dc_3(c_1+c_2)}}$$

$$C_A^\circ = \sqrt{\frac{c_1 c_2}{(c_1+c_2)} \frac{4D^2 c_s^2}{2\times 2 D c_s}}$$

$$C_A^\circ = \sqrt{\frac{c_1 c_2}{c_1+c_2} 2 D c_s}$$

$$C_A^\circ = \sqrt{\frac{2 D c_s c_1 c_2}{c_1+c_2}}$$

$$\boxed{C_A^\circ = \sqrt{2 D c_s c_1}\sqrt{\frac{c_2}{c_1+c_2}}}$$